

Pure patterns of order 2

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Abstract

We provide mutual elementary recursive order isomorphisms between classical ordinal notations, based on Skolem hulling, and notations from pure elementary patterns of resemblance of order 2, showing that the latter characterize the proof-theoretic ordinal 1^∞ of the fragment $\Pi_1^1\text{-CA}_0$ of second order number theory, or equivalently the set theory $\text{KP}\ell_0$. As a corollary, we prove that Carlson’s result on the well-quasi orderedness of respecting forests of order 2 implies transfinite induction up to the ordinal 1^∞ . We expect that our approach will facilitate analysis of more powerful systems of patterns.

1 Introduction

Elementary patterns of resemblance were discovered and then systematically introduced by Timothy J. Carlson, [2, 3, 4], as an alternative approach to recursive systems of ordinal notations. Elementary patterns constitute the basic levels of Carlson’s programmatic approach, *patterns of embeddings*, which is inspired by Gödel’s program of using large cardinals to solve mathematical incompleteness, cf. [8, 9]. It follows heuristics that axioms of infinity are in close correspondence with ordinal notations. The long-term goal of patterns of embeddings is therefore to find an ultra-finestructure for large cardinal axioms based on embeddings, thereby ultimately complementing inner model theory.

Patterns of resemblance, which instead of involving codings of embeddings, rely upon binary relations coding the property of elementary substructure of increasing complexity, are first steps to investigate patterns. Inspired by the notion of elementary substructure along ordinals as set-theoretic objects, ordinal notations in terms of elementary patterns intrinsically carry semantic content. However, Carlson made the intriguing observation that patterns have simple, finitely combinatorial characterizations called *respecting forests*.

The present article focuses on elementary patterns of order 2. Recalling from the introduction to [14], let $\mathcal{R}_2 = (\text{Ord}; \leq, \leq_1, \leq_2)$ be the structure of ordinals with standard linear ordering \leq and partial orderings \leq_1 and \leq_2 , simultaneously defined by induction on β in

$$\alpha \leq_i \beta :\Leftrightarrow (\alpha; \leq, \leq_1, \leq_2) \preceq_{\Sigma_i} (\beta; \leq, \leq_1, \leq_2)$$

where \preceq_{Σ_i} is the usual notion of Σ_i -elementary substructure (without bounded quantification), see [1, 3] for fundamentals and groundwork on elementary patterns of resemblance. Pure patterns of order 2 are the finite isomorphism types of \mathcal{R}_2 . The *core* of \mathcal{R}_2 consists of the union of *isominimal realizations* of these patterns within \mathcal{R}_2 , where a finite substructure of \mathcal{R}_2 is called isominimal, if it is pointwise minimal (with respect to increasing enumerations) among all substructures of \mathcal{R}_2 isomorphic to it, and where an isominimal substructure of \mathcal{R}_2 realizes a pattern P , if it is isomorphic to P . It is a basic observation, cf. [3], that the class of pure patterns of order 2 is contained in the class \mathcal{RF}_2 of *respecting forests of order 2*: finite structures P over the language (\leq_0, \leq_1, \leq_2) where \leq_0 is a linear ordering and \leq_1, \leq_2 are forests such that $\leq_2 \subseteq \leq_1 \subseteq \leq_0$ and \leq_{i+1} *respects* \leq_i , i.e. $p \leq_i q \leq_i r$ & $p \leq_{i+1} r$ implies $p \leq_{i+1} q$ for all $p, q, r \in P$, for $i = 0, 1$.

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In [7] we showed that every pattern has a cover below 1^∞ , the least such ordinal. As outlined in [14], an order isomorphism (embedding) is a cover (covering, respectively) if it maintains the relations \leq_1 and \leq_2 . The ordinal of $\text{KP}\ell_0$, which axiomatizes limits of models of Kripke-Platek set theory with infinity, is therefore least such that there exist arbitrarily long finite \leq_2 -chains. Moreover, by determination of enumeration functions of (relativized) connectivity components of \leq_1 and \leq_2 , we were able to describe these relations in terms of classical ordinal notations. The central observation in connection with this is that every ordinal below 1^∞ is the greatest element in a \leq_1 -chain in which \leq_1 - and \leq_2 -chains alternate, thus providing a formalism that allows for precise localization of ordinals in terms of relativized connectivity components of the relations \leq_1 and \leq_2 . We called such chains *tracking chains*, as they provide all \leq_2 -predecessors and the greatest \leq_1 -predecessors insofar as they exist.

In [14] we showed that the arithmetical characterization of the structure \mathcal{R}_2 up to the ordinal 1^∞ , which we denoted as \mathcal{C}_2 , is an elementary recursive structure. This guarantees the elementary recursiveness of the order isomorphisms between hull and pattern notations given here.

From these preparations we devise here an algorithm that assigns an isominimal realization within \mathcal{C}_2 to each respecting forest of order 2, thereby showing that each such respecting forest is in fact (up to isomorphism) a pure pattern of order 2. It turns out that isominimal realizations are pointwise minimal among all covers of the given forest. We therefore derive a method that calculates ordinals coded in pattern notations in terms of familiar hull notations, cf. [11].

The notion of closure introduced here further allows us to provide pattern notations for finite sets of ordinals below 1^∞ . We are going to define an elementary recursive function that assigns describing patterns $P(\alpha)$ to ordinals $\alpha \in 1^\infty$. Recalling again from [14], a descriptive pattern for an ordinal α is a pattern, the isominimal realization of which contains α . Descriptive patterns are given in a way that makes a canonical choice for normal forms, since in contrast to the situation in \mathcal{R}_1^+ , cf. [13, 6], there is no unique notion of normal form in \mathcal{R}_2 . The chosen normal forms are of least possible cardinality.

The mutual order isomorphisms between hull and pattern notations in the present article enable classification of a new independence result for $\text{KP}\ell_0$, as was already announced [14]. We demonstrate that Carlson's result in [5], according to which the collection of respecting forests of order 2 is well-quasi-ordered with respect to coverings, cannot be proven in $\text{KP}\ell_0$ or, equivalently, in the restriction $\Pi_1^1\text{-CA}_0$ of second order number theory to Π_1^1 -comprehension and set induction. On the other hand, we know that transfinite induction up to the ordinal 1^∞ of $\text{KP}\ell_0$ suffices to show that every pattern is covered [7].

2 Preliminaries

For a general introduction to proof theory and ordinal notation systems, see Pohlers [10]. Classical notations based on Skolem hulling [10] that are used here were provided in [11] together with structural insights particularly useful in analysis of patterns of resemblance, first demonstrated in [12]. A summary of this toolkit can be found in [13], where the core of the structure \mathcal{R}_1^+ was analyzed. This was further enhanced in Sections 5 and 6 of [6].

This article builds upon the results, arithmetical tools, and terminology of [7] and [14]. The central notion is that of *tracking chains*, introduced in Definitions 5.1, 5.2, and 6.1 of [7], and thoroughly explained and analyzed in Section 5 of [14]. It provides a detailed description of the relations \leq_1 and \leq_2 in terms of (relativized) connectivity components, thereby providing “addresses” for the ordinals below 1^∞ in terms of nested components of \leq_i , $i = 1, 2$. Corollary 5.8 of [14] summarizes the arithmetical, and even syntactic, characterization of the semantic relations \leq_i , coding Σ_i -elementarity within \mathcal{R}_2 , up to 1^∞ .

Notions of closedness and closure introduced in the present article build upon the notion of (relativized) spanning sets of tracking chains, introduced in Definitions 5.1, 5.2, and 5.3 of [14].

3 Spanning and closed sets of tracking chains

The notion of closedness for sets of tracking chains is central to the investigation of the core of \mathcal{R}_2 , as it is crucial for isominimal realization. In preparation for a relativized notion of closedness, we will first introduce sets of tracking chains that are spanning above some given tracking chain α , considerably extending sets of tracking chains that are weakly spanning above α according to Definition 5.3 of [14]. We begin with some useful notation.

Definition 3.1 *Let $M \subseteq_{\text{fin}} \text{TC}$.*

1. For $\alpha \in M$ where $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,m_i})$, $1 \leq i \leq n$, let M_α denote the subset

$$M_\alpha := \{\beta \in M \mid o(\alpha) <_1 o(\beta)\}.$$

We also sometimes denote a finite set of $<_1$ -successors of some $\alpha \in \text{TC}$ by M_α , i.e. a superset M containing α is not required. For $\alpha \in \text{TC}$ we define $I(\alpha)$ to be the set of all initial chains of α , including α . For convenience we set $I(()) := \emptyset$ and $M_\emptyset := M$.

2. Set $\text{gs}(M_\alpha) := 0$ if $M_\alpha = \emptyset$, otherwise let $\beta \in M_\alpha$, $\beta_i = (\beta_{i,1}, \dots, \beta_{i,k_i})$, $1 \leq i \leq l$, be the unique chain corresponding to the greatest immediate $<_1$ -successor of $o(\alpha)$ in $o[M_\alpha]$, and let σ be the chain associated with β . We define

$$\text{gs}(M_\alpha) := \begin{cases} \beta_{l,1} & \text{if } k_l = 1 \\ \sigma_{l,k_l-1} & \text{otherwise,} \end{cases}$$

and call $\text{gs}(M_\alpha)$ the κ -index of the greatest immediate $<_1$ -successor of α in M_α .

We now strengthen the notion of weakly spanning sets of tracking chains above some α . Proposition 5.10 of [14] will play a central role in the definition of (relativized) spanningness as it characterizes \leq_1 in terms of tracking chains in the sense that necessary and sufficient conditions for tracking chains $\alpha, \beta \in \text{TC}$ to satisfy $o(\alpha) \leq_1 o(\beta)$ are given. Recall from [14], Lemma 5.11, that the relation \subseteq of initial chain on TC respects the ordering \leq_{TC} , hence also the characterization of \leq_1 on TC.

Definition 3.2 According to Proposition 5.10 of [14], for $\alpha, \beta \in \text{TC}$ such that $o(\alpha) <_1 o(\beta)$ and $\alpha \not\subseteq \beta$ there exists $(i, j) \in \text{dom}(\alpha) \cap \text{dom}(\beta)$ such that $\alpha_{\upharpoonright i,j} = \beta_{\upharpoonright i,j}$ and $\alpha_{i,j+1} < \beta_{i,j+1}$. We call $(i, j+1) =: \text{bp}(\alpha, \beta)$ the branching index pair of α and β . If the above conditions on α and β do not hold, we say that the branching index pair of α and β does not exist.

Definition 3.3 (Spanning sets of tracking chains above α) Let $\alpha \in \text{TC}$ and $M_\alpha \subseteq_{\text{fin}} \text{TC}$ be as in the above definition. M_α is called spanning above α if it is closed under clauses 1 – 6 of Definitions 5.1 and 5.2 of [14] with the modification that the resulting respective tracking chains β satisfy $o(\alpha) <_1 o(\beta)$, and if it

7. supplies implicit maximal extensions: For any $\beta \in M_\alpha$ such that $\text{bp}(\alpha, \beta) =: (i, j+1)$ exists with $\chi^{\tau_{i,j}}(\tau_{i,j+1}) = 1$ (where τ is the chain associated with α), we have $\text{me}(\alpha_{\upharpoonright i,j+1}) \in M_\alpha$, and
8. extends main lines: if $\text{cml}(\beta) =: (i, j)$ exists for some $\beta \in M_\alpha$, then $\beta_{\upharpoonright i,j+1}[\mu_{\sigma_{i,j}}] \in M_\alpha$ where σ is the chain associated with β .

For $\alpha = ()$ any spanning set of tracking chains according to Definition 5.2 of [14] is called spanning above α .

Remark. Any $M \subseteq_{\text{fin}} \text{TC}$ that is spanning according to Definition 5.2 of [14], is closed under clauses 7 and 8; hence closure under clauses 1 – 8 is a finite process.

Lemma 3.4 Let M be spanning above some $\alpha \in \text{TC}$. Then M is closed under me , lh , and lh_2 . If α is convex, then every $\beta \in M$ is a proper extension of α , i.e. $\alpha \subseteq \beta$. Thus, for convex α , M is spanning above α if and only if it is weakly spanning above α according to Definition 5.3 of [14].

Proof. Lemma 5.4 of [14] yields the claim regarding me . Condition 8 above in conjunction with condition 2 of Definition 5.1 and characterization (7) of lh in Section 5.3 of [14] then imply the claim regarding lh . And the claim regarding lh_2 follows from Corollary 5.9 of [14]. If α is convex, conditions 7 and 8 never apply to (i, j) such that $(i, j+1) \in \text{dom}(\alpha)$. \square

For given $\alpha < 1^\infty$, the following proposition characterizes the ordinals β such that $\alpha <_1 \beta$, and there does not exist any γ with $\alpha < \gamma <_2 \beta$ in terms of tracking chains, cf. Theorem 7.9 of [7] and Proposition 5.6 of [14].

Proposition 3.5 (Relative \leq_2 -minimality) Let $\alpha, \beta \in \text{TC}$ satisfy $\alpha := o(\alpha) <_1 o(\beta) =: \beta$. Let σ be the chain associated with β , $\beta_i = (\beta_{i,1}, \dots, \beta_{i,k_i})$, $i = 1, \dots, l$. According to Theorem 7.9 of [7] β is \leq_2 -minimal if and only if $k_l \leq 2$ and $\sigma_l^* = 1$. In the case $\alpha \not\subseteq \beta$ let $(i, j+1) =: \text{bp}(\alpha, \beta)$, otherwise let $(i, j) = (n, m_n)$. Then

$$\beta \text{ is } \alpha\text{-}\leq_2\text{-minimal if and only if either } (i, j+1) = (l, k_l) \text{ or } k_l \leq 2 \ \& \ l^* <_{\text{lex}} (i, j).$$

Proof. Cf. Theorem 7.9 of [7] and Proposition 5.6 of [14]. \square

The following definition and theorem give a flavor of the expressive power of tracking chains in the sense that isomorphisms of intervals with the same $<_2$ -predecessors can be identified easily.

Definition 3.6 (Vertical translation) Let $\alpha \in \text{TC} \cup \{()\}$, where $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,m_i})$, $1 \leq i \leq n$, $n \geq 0$. Let $M = M_\alpha \subseteq_{\text{fin}} \text{TC}$ be a set of proper extensions of α of the form $M = \{\beta\} \cup M_\beta$ for a tracking chain β , where $\beta_i = (\beta_{i,1}, \dots, \beta_{i,k_i})$, $1 \leq i \leq l$, such that $k_l = 1$, $l^* = (1, 0)$ if $\alpha = ()$, and $l^* <_{\text{lex}} (n, m_n)$ otherwise, so that each $\gamma \in M$ is an extension of β . For $\gamma \in M$ we define the tracking chain γ' by

$$\gamma'_{i,j} := \begin{cases} \alpha_{i,j} & \text{if } (i, j) \in \text{dom}(\alpha) \\ \tau_{l,1} & \text{if } (i, j) = (n+1, 1) \\ \gamma_{l-n-1+i,j} & \text{if } (n+1, 1) <_{\text{lex}} (i, j) \text{ \& } (l-n-1+i, j) \in \text{dom}(\gamma). \end{cases} \quad (1)$$

We define $M' := \{\gamma' \mid \gamma \in M\}$.

Theorem 3.7 Let M, α, β be as in the above definition and $I := I(\alpha)$ as in Definition 3.1. Then M' consists of tracking chains that properly extend α , and the images $\text{o}[I \cup M]$ and $\text{o}[I \cup M']$ are isomorphic substructures of C_2 , both closed under $<_2$ -predecessors.

Proof. The claims are verified by close inspection of the definitions involved. Notice that $\beta' = \alpha \frown (\tau_{l,1})$ is a tracking chain since our assumptions prevent a violation of condition 5 in Definition 5.1 of [7] and imply that for all $(r, s) \in \text{dom}(\beta)$ such that $l^* =: (i, j) <_{\text{lex}} (r, s) <_{\text{lex}} (l, 1)$ we have $\tau_{l,1} < \rho_r(\beta_{\upharpoonright r,s})$. \square

Remark. Note that in the case where $\text{cml}(\beta) =: (i, j)$ exists, the isomorphic copy M' might lose $<_1$ -connections up to $\beta^+ := \beta_{\upharpoonright i,j+1}[\mu_{\tau_{i,j}}]$.

Definition 3.8 Let $M \subseteq_{\text{fin}} \text{TC}$. We will make use of the notation M_α as in Definition 3.1.

1. Suppose $\alpha \in \text{TC}$, $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,m_i})$, $1 \leq i \leq n$, such that $m_n > 1$ and $\alpha_{n,m_n} = \mu_\tau$, where $\tau = \tau_{n,m_n-1}$, τ denoting the chain associated with α . Then α is called a principal chain (to base τ), and τ is called the base of α . If $\alpha \in M$ then we say that α is a principal chain in M and that τ is a base in M .
2. Let α be as in part 1 and $\beta \in M_\alpha$, where $\beta_i = (\beta_{i,1}, \dots, \beta_{i,k_i})$, $1 \leq i \leq l$, with associated chain τ . If $\alpha \subseteq \beta$ let $r \in (n, l]$ be minimal such that $\tau \not\vdash \beta_{r,1}$, i.e. $\tau_{r,1} < \tau$, and $\tau_r^* < \tau_{r,1}$, if that exists. Otherwise set $r := 0$. Then $\text{pi}_{M_\alpha}(\beta) := r$ is called the parameter index of β in M_α , and the parameter of β in M_α is defined by $\text{par}_{M_\alpha}(\beta) := \tau_{r,1}$ if $r > 0$ and $\text{par}_{M_\alpha}(\beta) := 0$ otherwise. We will omit the subscript M_α when this context is unambiguous. The set of parameters of M_α is then defined by

$$\text{par}(M_\alpha) := \{\text{par}(\beta) \mid \beta \in M_\alpha\},$$

and its maximum is denoted by $\text{mp}(M_\alpha) := \max(\text{par}(M_\alpha))$.

3. Suppose in addition to the assumptions of part 2 that M_α is spanning above α and that β is the tracking chain of $\max(\text{o}[M_\alpha])$, i.e. β is the $<_{\text{TC}}$ -maximum of M_α . In the case where

$$\max\{\text{par}(\gamma) \mid \gamma \in M_\alpha \text{ \& } \gamma \not\subseteq \beta\} < \text{par}(\beta) \in \mathbb{E}$$

and either α is convex or $\text{lh}(\text{o}(\alpha)) = \text{o}(\beta)$, we call $\text{db}(M_\alpha) := \text{par}(\beta)$ the distinguished base of M_α and $\text{dc}(M_\alpha) := \beta_{\upharpoonright r,1}$ (where $r = \text{pi}(\beta)$) the distinguished chain in M_α . In all other cases we set $\text{db}(M_\alpha) := 0$ and $\text{dc}(M_\alpha) := ()$.

4. Let M_α and β be as in part 3 and suppose that $\sigma := \text{db}(M_\alpha) > 0$. If $(n, m_n + 1) \in \text{dom}(\tau)$, i.e. $m_n < k_n$, define $\sigma_0 := (\tau_{n,m_n}, \dots, \tau_{n,k_n-1})$, otherwise set $\sigma_0 := ()$. Then define $\sigma_j := (\tau_{n+j,1}, \dots, \tau_{n+j,k_{n+j}-1})$ for $j = 1, \dots, r - n - 1$, and $\sigma_{r-n} := (\sigma)$. Finally define σ to be the concatenation of the vectors σ_j , $j = 0, \dots, r - n$. The distinguished sequence of M_α is then defined by $\text{ds}(M_\alpha) := \sigma$, and in all cases where the above conditions are not met we set $\text{ds}(M_\alpha) := ()$.

Recall the operator $\bar{\cdot}$ from Section 8 of [11] and Section 5 of [6].

Definition 3.9 (Closedness) Let $M \subseteq_{\text{fin}} \text{TC}$ be spanning (spanning above α). M is closed (closed above α) if and only if for all principal chains β in M such that $\bar{\tau} \in (\tau', \tau)$, where τ is the base of β and τ denotes the chain associated with β , we have

$$\text{mp}(M_\beta) \begin{cases} \geq \bar{\tau} & \text{if } \text{db}(M_\beta) = 0 \\ > \bar{\tau} & \text{otherwise.} \end{cases} \quad (2)$$

We call the base τ of a principal chain β in M such that $\bar{\tau} \in (\tau', \tau)$ a supported base in M if and only if (2) holds, otherwise we call τ a non-supported base in M .

Lemma 3.10 Let M be closed (closed above α). Then M is closed under $\bar{\cdot}$ (above α) in the following sense: for any principal chain $\beta \in M$ with supported base $\tau = \tau_{i,j}$ as in the above definition there exists a principal chain $\gamma \in M_\beta$, $\beta \subseteq \gamma$, to base $\bar{\tau} = \tau_{r,s}$ such that the bases $\tau' = \tau'_{i,j}$ and $\bar{\tau}' = \tau'_{r,s}$ have the same index pair $(i, j)' = (r, s)'$.

Proof. The claim follows from closedness by induction on the height of M_β , using Lemma 5.9 of [6]. \square

Definition 3.11 (Closure) Let $\alpha \in \text{TC} \cup \{()\}$ and $M = M_\alpha \subseteq_{\text{fin}} \text{TC}$ be a set of tracking chains as in Definition 3.1. The closure of M above α , denoted as M^{cl} , is the least set of tracking chains containing M that is closed under clauses 1 – 8, relaxed by the condition that in the case $\alpha \neq ()$ the respective resulting tracking chains β satisfy $\text{o}(\alpha) <_1 \text{o}(\beta)$, cf. Definitions 5.1 and 5.2 of [14] and Definition 3.3, and that

9. supports bases: if β is a principal chain in M to base τ such that $\bar{\tau} \in (\tau', \tau)$ then $\beta^\frown(\bar{\tau}) \in M$, unless condition 2 of Definition 3.9 holds anyway.

Remark. Notice that the above clause for base support makes a choice in the support of bases. The process of closure is finite since application of the operator $\bar{\cdot}$ strictly lowers the l-measure, see inequality 3 following Definition 4.3 of [14].

Definition 3.12 (Essential closedness) Let $M \subseteq_{\text{fin}} \text{TC}$ be spanning (spanning above α). M is essentially closed (above α) if and only if the closure \bar{M} of M under initial chains (β such that $\text{o}(\alpha) <_1 \text{o}(\beta)$) is closed (closed above α) and only adds tracking chains of a form $\beta^\frown(\gamma_1)$ where $\beta^\frown(\gamma_1, \gamma_2) \in M$ for some γ_2 such as $\mu_{\text{end}(\gamma_1)}$.

Remark. Essentially closed sets remain to be closed under me, lh, and lh₂ in the sense of Corollaries 5.13 and 5.9 of [14].

The following definition of essential closure of a given set M of tracking chains allows us to omit redundant chains. Such chains do not belong to the original set M , end in a κ -index, and have 1-step extensions in M^{cl} , but only by ν -indices.

Definition 3.13 (Essential closure) Let $\alpha \in \text{TC} \cup \{()\}$ and $M = M_\alpha \subseteq_{\text{fin}} \text{TC}$ be a set of tracking chains as in Definition 3.1. The essential closure of M above α , denoted as M^{eccl} , is obtained from M^{cl} by dropping all tracking chains $\gamma \in M^{\text{cl}} - M$ that are of a form $\beta^\frown(\gamma_1)$ where $\beta^\frown(\gamma_1, \gamma_2) \in M^{\text{cl}}$ for some γ_2 and for which there does not exist any proper extension of a form $\gamma^\frown\gamma' \in M^{\text{cl}}$.

We are now prepared to introduce the notions of κ -index and base minimization. These provide the key tools in the algorithm that assigns isominimal realizations to given respecting forests of order 2 by determining minimal (relativized) \leq_1 - and \leq_2 -components, respectively, that satisfy a given forest.

Definition 3.14 (κ -index minimization) Let α be either the empty sequence or a convex tracking chain, where $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,m_i})$, $1 \leq i \leq n$, $n \geq 0$. Let $M = M_\alpha \subseteq_{\text{fin}} \text{TC}$ be a set of proper extensions of α of the form $M = \{\beta\} \cup M_\beta$ for a convex tracking chain β with associated chain τ such that M_β is either empty or closed above β and either

1. $\beta = \alpha^\frown(\beta_{n+1,1})$, where we set $\tau := \tau_{n+1,1}$, or
2. $\beta = \alpha^\frown(\beta_{n+1,1}, \mu_\tau)$ where $\tau := \tau_{n+1,1}$ or
3. β extends α by the ν -index $\beta_{n,m_n+1} = \mu_\tau$ where $\tau := \tau_{n,m_n}$, $n > 0$.

Set $\xi := \text{gs}(M_\beta)$ and suppose σ to be either the base of a $<_2$ -predecessor γ of $\beta := \text{o}(\beta)$, $\gamma := \text{tc}(\gamma)$, or $\sigma = 1$ and $\gamma = 0$, $\gamma := ()$, such that all $<_2$ -predecessors $\delta \leq \beta$ of ordinals in $\text{o}[M_\beta]$ satisfy $\delta \leq \gamma$. We call γ the chain of the preserved $<_2$ -predecessor and σ its base. Note that σ and γ determine each other and that according to the assumptions β does not have any $<_2$ -successor in $\text{o}[M_\beta]$. Setting $\eta := 0$ in the case $\alpha = ()$ & $M_\beta = \emptyset$, and $\eta := \sigma \cdot \omega^\xi$ otherwise, if β is of the form 1 we then have $\sigma \mid \beta_{n+1,1}$ and, moreover, $\eta \leq \beta_{n+1,1} < \rho_n$ since $\sigma \leq \tau_{n+1}^*$, and if β is of the form 2 or 3 we have $\xi < \tau$ and hence $\eta < \tau < \rho_n$.

We define the κ -index minimization above σ in M at β , denoted as $\kappa_{M,\beta,\sigma}$, or equivalently the κ -index minimization in M at β preserving γ , denoted as $\kappa_{M,\beta,\gamma}$, and κ in short, as follows.

$$\kappa(\beta) := \alpha^\frown(\eta) \quad \text{and} \quad \kappa\text{-idx} := \eta,$$

and for $\delta \in M_\beta$ we define $\kappa(\delta)$ by considering the following cases.

Case 1: $\beta = \alpha^\frown(\beta_{n+1,1})$ and $\xi = \tau_{n+1,1} \in \mathbb{E}^{>\tau_{n+1}^*}$. Then we only change the index $\beta_{n+1,1}$ at $(n+1, 1)$ in δ to $\tau_{n+1,1}$ in order to obtain $\kappa(\delta)$, which we call a horizontal translation.

Case 2: Otherwise. Then we have $\xi < \tau$.

Subcase 2.1: $\xi = \eta$ and $\beta^\frown(\xi) \subseteq \delta$. Then we define for $\delta = \beta^\frown(\xi)^\frown\delta'$

$$\kappa(\delta) := \alpha^\frown(\eta, 1)^\frown\delta',$$

and for δ of a form $\beta^\frown(\xi, \delta_{n+2,2}, \dots, \delta_{n+2,k_{n+2}})^\frown\delta'$ we define

$$\kappa(\delta) := \alpha^\frown(\eta, 1 + \delta_{n+2,2}, \delta_{n+2,3}, \dots, \delta_{n+2,k_{n+2}})^\frown\delta'.$$

Subcase 2.2: Otherwise. Then we simply replace the initial sequence β of δ by $\alpha^\frown(\eta)$ in order to obtain $\kappa(\delta)$, i.e., writing δ in the form $\beta^\frown\delta'$ we define

$$\kappa(\delta) := \alpha^\frown(\eta)^\frown\delta'.$$

Theorem 3.15 Let M, α, β and σ, γ be as in the above definition as well as the shortcuts β, γ , and set $\alpha := \text{o}(\alpha)$, $\beta_\kappa := \text{o}(\kappa(\beta))$, and $I := I(\gamma)$. Then $\kappa[M]$ is a set of tracking chains and we have

1. $\alpha <_1 \text{o} \circ \kappa[M]$,
2. $\text{Pred}_2(\beta_\kappa) = \{\delta \mid \delta \leq_2 \gamma\}$, if $\gamma \neq 0$, otherwise β_κ is \leq_2 -minimal.
3. the images of $I \cup M$ and $I \cup \kappa[M]$ under o are isomorphic substructures of \mathcal{C}_2 , and
4. $\kappa[M_\beta]$ is closed above $\kappa(\beta)$.
5. $\kappa[M]$ is closed above α .

Proof. The theorem directly follows from the definitions involved. □

In order to formulate the notion of base minimization in sets of tracking chains, see Definition 3.20, we need to make a few important technical observations.

Lemma 3.16 Let M_α and β be as in part 4 of Definition 3.8. Setting $\xi := \text{gs}(M_\alpha)$ and $\sigma := \text{ds}(M_\alpha)$ where $\sigma = (\sigma_1, \dots, \sigma_s)$ we have

$$\text{end}(\xi) = \sigma_1.$$

The sequence σ is obtained from ξ by the exhaustive, uniquely determined iterated procedure of taking the last summand (i.e. applying end) and then applying either the λ -operator in the case of an epsilon number or otherwise the operator $\rho \mapsto \log((1/\rho') \cdot \rho)$.

Proof. This is a consequence of Definition 5.2 of [14] and the preceding remark there regarding intermediate ν -indices which are obtained by application of the λ -operator. □

Recall Definitions 4.11 of [11] and 3.1 of [7]. We are going to trace the same occurrence of the particular subterm checked by the indicator function χ^τ for being τ .

Definition 3.17 For $\tau \in \mathbb{E}$ and $\xi \in T^\tau$ we define a sequence $\text{mq}^\tau(\xi)$ of subterms by

1. $\text{mq}^\tau(\xi) := (\xi)$ if $\xi \leq \tau$
2. $\text{mq}^\tau(\xi) := (\xi) \smallfrown \text{mq}^\tau(\eta)$ if $\xi =_{\text{NF}} \delta + \eta > \tau$
3. For $i < \omega$ and $\xi = \vartheta_i(\Delta + \eta)$ where $\eta < \Omega_{i+1} \mid \Delta$ with $\Delta + \eta > 0$ in case of $i = 0$ we define
 - 3.1. $\text{mq}^\tau(\xi) := (\xi) \smallfrown \text{mq}^\tau(\Delta)$ if $\zeta_\xi^\tau = 0$
 - 3.2. $\text{mq}^\tau(\xi) := (\xi) \smallfrown \text{mq}^\tau(\eta)$ otherwise.

Lemma 3.18 Let M_α and β be as in part 4 of Definition 3.8. Setting $\xi := \text{gs}(M_\alpha)$, the sequence $\text{mq}^\tau(\xi)$ ends in σ , and the sequence $\sigma := \text{ds}(M_\alpha)$ characterizes the same occurrence of σ as a subterm of ξ .

Proof. Term decompositions carried out to obtain σ and $\text{mq}^\tau(\xi)$, respectively, only differ inessentially when the terminal element is an epsilon number. The main difference is the application of ι -operators (as components of λ -operators) in σ , which carry out a straightforward exchange of ϑ -functions. \square

Recall the notation $\alpha^\mathbb{E}$ introduced in Section 2 of [11] for the least epsilon number strictly greater than α . The following lemma provides a crucial estimation of the term parameters, cf. Definition 3.28 of [11], in closed sets of tracking chains.

Lemma 3.19 Let β be a convex principal chain to base τ , with associated chain τ , and let $M = M_\beta$ be closed above β . Then for all $\gamma \in M$ and all $(i, j) \in \text{dom}(\gamma) - \text{dom}(\beta)$ such that either $r := \text{pi}_M(\gamma) = 0$ or $(i, j) \leq_{\text{lex}} (r, 1)$ we have

$$\text{Par}^\tau(\gamma_{i,j}) \subseteq \text{mp}(M)^\mathbb{E}.$$

Proof. In the notation of part 1 of Definition 3.8 we have $\tau = \tau_{l,k_l-1}$; thus the setting of relativization of M is given by $\sigma_0 := \text{cs}(\beta_{\upharpoonright l,k_l-1}) \in \text{RS}$. The indices $\gamma_{i,j}$ can therefore be considered as elements of T^{σ_0} , where $\sigma_0 \subseteq \sigma$ is according to the nestings of ι -operators involved in the application of μ - and λ -operators, see Definition 4.3 and subsequent remark of [14]. The lemma now follows by induction on $l^\sigma(\gamma_{i,j})$ for the appropriate extension σ of σ_0 , since for epsilon numbers γ we have $\text{Par}^\tau(\gamma) \subseteq \text{Par}^\tau(\bar{\gamma}) \cup \text{Par}^\tau(\lambda_\gamma)$, and for ordinals of a form $\gamma = \vartheta^\sigma(\eta) > \tau$ where $\eta < \Omega_1$, i.e. which are not epsilon numbers, we have $\text{Par}^\tau(\gamma) = \text{Par}^\tau(\eta)$, cf. Equation 1 of [14]. \square

We now turn to base minimization in sets of tracking chains. This provides a tool to determine \leq_{pw} -minimal isomorphic copies of sets of tracking chains. Recall the notion of base transformation, cf. Section 5 of [11] or in short Definition 2.15 of [13]. For convenience we set $\pi_{\tau,\tau} := \text{id}$.

Definition 3.20 (Base minimization) Let α be either the empty sequence or a convex tracking chain, where $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,m_i})$, $1 \leq i \leq n$, $n \geq 0$. Let $M = M_\alpha \subseteq_{\text{fin}} \text{TC}$ be a set of proper extensions of α of a form $M = \{\beta\} \cup M_\beta$, where $\text{o}[M_\beta]$ contains a $<_2$ -successor of $\text{o}(\beta)$, M_β is closed above β , and β is a convex principal chain in M to base τ , consisting of the vectors $\beta_i = (\beta_{i,1}, \dots, \beta_{i,k_i})$, $1 \leq i \leq l$, such that either

1. $\beta = \alpha \smallfrown (\beta_{n+1,1}, \mu_\tau)$ with $\beta_{n+1,1} =_{\text{NF}} \beta' + \tau$ or
2. β extends $\alpha \neq ()$ by the ν -index $\beta_{n,m_n+1} = \mu_\tau$, $\beta' := 0$.

Set $\alpha := \text{o}(\alpha)$, $\beta := \text{o}(\beta)$, $\xi := \text{gs}(M_\beta) \geq \tau$, $\sigma_0 := \text{db}(M_\beta)$, and $\sigma_1 := \max\{\text{mp}(M_\beta), \rho\}$, where ρ is either the base of a $<_2$ -predecessor δ of β , setting $\delta := \text{tc}(\delta)$, or $\rho = 1$, setting $\delta := 0$ and $\delta := ()$, such that all greatest $<_2$ -predecessors $\gamma < \beta$ of ordinals in $\text{o}[M_\beta]$ satisfy $\gamma \leq \delta$. We call δ the chain of the preserved $<_2$ -predecessor and ρ its base. Note that ρ and δ determine each other.

We define the base minimization above ρ in M at β , $\pi_{M,\beta,\rho}$, or equivalently the base minimization in M at β preserving δ , $\pi_{M,\beta,\delta}$, as follows, where we simply write π , whenever the arguments M, β , and ρ or δ are understood from the context. In order to define $\pi(\gamma)$ for $\gamma \in \{\beta\} \cup M_\beta$ we consider the following cases.

Case 1: $\sigma_0 \leq \rho$ or otherwise $\pi_{\sigma_0,\tau}^{-1}(\lambda_{\sigma_0}) < \xi$. Let $\sigma \in \mathbb{E} \cap (\sigma_1, \tau]$ be minimal such that $\xi \leq \pi_{\sigma,\tau}^{-1}(\lambda_\sigma)$. Minimality of σ then implies that $\xi = \pi_{\sigma,\tau}^{-1}(\lambda_\sigma)$, see Lemmata 5.8 and 8.2 of [11]. In the case $\sigma = \tau$ transformation to a smaller base is not possible, and if assumption 2 holds for β then we set $\pi := \text{id}$. Otherwise define

$$\pi(\beta) := \alpha \smallfrown (\sigma, \mu_\sigma),$$

and for $\gamma \in M_\beta$ and $r := \text{pi}_{M_\beta}(\gamma)$ we either have $r = 0$ or $r > l$ and define

$$\pi(\gamma)_{i,j} := \begin{cases} \pi(\beta)_{i,j} & \text{if } (i,j) \in \text{dom}(\beta) \\ \gamma_{i,j} & \text{if } r > 0 \text{ \& } (r,1) <_{\text{lex}} (i,j) \\ \pi_{\sigma,\tau}(\gamma_{i,j}) & \text{otherwise,} \end{cases} \quad (3)$$

which in the case $\sigma = \tau$ & $\tau < \beta_{n+1,1}$ performs a horizontal translation.

Case 2: $\sigma := \sigma_0 > \rho$ and $\xi \leq \pi_{\sigma,\tau}^{-1}(\lambda_\sigma)$.

Subcase 2.1: $\tau \nmid \xi$. Then, due to the uniqueness of $\sigma = \text{db}(M_\beta)$, we have $\xi = \tau \cdot \nu + \sigma$ for some $\nu > 0$, which we write as $\nu = \lambda + k \dot{-} \chi^\tau(\lambda)$, where $\lambda \in \text{Lim} \cup \{0\}$ and $k < \omega$ such that if $\chi^\tau(\lambda) = 1$ & $\nu = \lambda$ then $k = 1$. Therefore $\pi_{\sigma,\tau}(\tau \cdot \nu) = \sigma \cdot (\pi_{\sigma,\tau}(\lambda) + k \dot{-} \chi^\sigma(\pi_{\sigma,\tau}(\lambda)))$, and setting $\delta := \omega^{\pi_{\sigma,\tau}(\lambda)+k}$ we obtain $\xi = \pi_{\sigma,\tau}^{-1}(\varrho_\delta^\sigma) + \sigma \leq \lambda_\tau$ due to part 2 of Lemma 2.3 of [14] and $\chi^\tau(\omega^{\lambda+k}) = 0$. Define

$$\pi(\beta) := \alpha^\frown(\sigma, \delta),$$

and for $\gamma \in M_\beta$ such that $\gamma <_{\text{TC}} \beta^\frown(\xi)$ we define $\pi(\gamma)$ as in (3) of Case 1. For $\gamma \in M_\beta$ such that $\beta^\frown(\xi) \leq_{\text{TC}} \gamma$, which we may write as $\gamma = \beta^\frown(\xi, \gamma_{l+1,2}, \dots, \gamma_{l+1,k_{l+1}})^\frown \gamma'$, we define

$$\pi(\gamma) := \begin{cases} \alpha^\frown(\sigma, \delta + 1)^\frown \gamma' & \text{if } k_{l+1} = 1 \\ \alpha^\frown(\sigma, \delta + 1 + \gamma_{l+1,2}, \gamma_{l+1,3}, \dots, \gamma_{l+1,k_{l+1}})^\frown \gamma' & \text{otherwise.} \end{cases}$$

Subcase 2.2: $\tau \mid \xi$. We then have $\xi = \tau \cdot \nu$ for some $\nu > 0$ which we write as $\nu = \lambda + k \dot{-} \chi^\tau(\lambda)$ where $\lambda \in \text{Lim}$ and $k < \omega$. According to the definition of σ we have $k \dot{-} \chi^\tau(\lambda) = 0$. Lemma 3.18 shows that $\chi^\tau(\lambda) = 0$ and hence $k = 0$. According to our assumptions σ has a unique occurrence in λ and $\max(\text{Par}^\tau(\lambda)) = \sigma$, and we may apply $\pi_{\sigma,\tau}$ to λ , simply leaving σ unchanged, thus obtaining $\chi^\sigma(\pi_{\sigma,\tau}(\lambda)) = 1$. We now have $\pi_{\sigma,\tau}(\xi) = \sigma \cdot \pi_{\sigma,\tau}(\lambda)$ and set $\delta := \omega^{\pi_{\sigma,\tau}(\lambda)}$, so that $\chi^\sigma(\delta) = 1$. Define

$$\pi(\beta) := \alpha^\frown(\sigma, \delta),$$

and for $\gamma \in M_\beta$ such that $\gamma <_{\text{TC}} \text{dc}(M_\beta)$ we define $\pi(\gamma)$ again as in (3) of Case 1. For $\gamma \in M_\beta$ such that $\text{dc}(M_\beta) \leq_{\text{TC}} \gamma$, which, setting $r := \text{pi}_{M_\beta}(\max(M_\beta)) = \text{pi}_{M_\beta}(\gamma)$, we may write as $\gamma = \gamma_{\upharpoonright_{r-1}}^\frown(\gamma_{r,1}, \dots, \gamma_{r,k_r})^\frown \gamma'$, we define

$$\pi(\gamma) := \begin{cases} \alpha^\frown(\sigma, \delta + 1)^\frown \gamma' & \text{if } k_r = 1 \\ \alpha^\frown(\sigma, \delta + 1 + \gamma_{r,2}, \gamma_{r,3}, \dots, \gamma_{r,k_r})^\frown \gamma' & \text{otherwise.} \end{cases}$$

This concludes the definition of $\pi = \pi_{M,\beta,\rho} = \pi_{M,\beta,\delta}$, and for convenience we introduce the notations

$$\pi\text{-idx} := \sigma$$

and

$$\alpha_\pi^+ := \alpha^\frown(\sigma),$$

unless we have $\sigma = \tau$ in assumption 2 for β , where we set $\alpha_\pi^+ := \pi(\beta)$.

Theorem 3.21 Let M, α, β , and ρ, δ be as in the above definition as well as the shortcuts $\alpha, \beta, \delta, \sigma, \tau$, and set $I := I(\delta)$ and $\beta_\pi := \text{o}(\pi(\beta))$. Then $\pi[M]$ is a set of tracking chains, and we have

1. $\alpha <_1 \text{o} \circ \pi[M]$,
2. $\text{Pred}_2(\beta_\pi) = \{\gamma \mid \gamma \leq_2 \delta\}$ if $\delta \neq 0$, otherwise β_π is \leq_2 -minimal,
3. the images of $I \cup M$ and $I \cup \pi[M]$ under o are isomorphic substructures of \mathcal{C}_2 ,
4. $\pi[M_\beta]$ is closed above $\pi(\beta)$, and
5. $\{\alpha_\pi^+\} \cup \pi[M]$ is closed above α , hence $\pi[M]$ is essentially closed above α .

Proof. Due to Lemma 3.19, all terms to which the order preserving base transformation $\pi_{\sigma,\tau}$ is applied, use parameters below σ (with the unique exception handled explicitly in Subcase 2.2) and can be translated into T^τ , cf. Section 6 of [11], invariantly regarding localization (Lemma 6.5 of [11]), the operator $\bar{\cdot}$ and fine-localization (Lemma 5.7 of [6]), the operators ζ, λ, μ (Lemmata 6.8 and 7.7 of [11] and Lemma 3.6 of [7]), hence also regarding tracking sequences. We have verified commutativity of $\pi_{\sigma,\tau}$ with ζ, λ, μ (Lemmata 5.6, 7.10 of [11] and Lemma 3.7 of [7]), with $\bar{\cdot}$ (Lemma 5.7 of [6]), and also with the indicator χ and the operator ϱ (Lemmata 3.2 and 3.11 of [7]). For χ and ϱ , however, we need full commutativity with $\pi_{\sigma,\tau}$ with respect to the base argument as well, i.e.

$$\chi^\gamma(\eta) = \chi^{\pi_{\sigma,\tau}(\gamma)}(\pi_{\sigma,\tau}(\eta)) \quad \text{and} \quad \pi_{\sigma,\tau}(\varrho_\eta^\gamma) = \varrho_{\pi_{\sigma,\tau}(\eta)}^{\pi_{\sigma,\tau}(\gamma)}$$

for suitable arguments γ and η . For χ this property obviously holds; hence it also follows for ϱ . Inspecting the translation mapping we also observe that

$$\pi_{\sigma,\tau}(\vartheta^\gamma(\eta)) = \vartheta^{\pi_{\sigma,\tau}(\gamma)}(\pi_{\sigma,\tau}(\eta))$$

for suitable arguments γ and η . Commutativity of $\pi_{\sigma,\tau}$ with addition, multiplication, ω -exponentiation, and log is obvious. Therefore $\pi_{\sigma,\tau}$ also commutes with maximal (1-step) extensions (me), cf. Definition 5.2 of [7]. Close inspection of the definition of π now shows that $\pi[M] \subseteq \text{TC}$ and that $I \cup \pi[M]$ is isomorphic to $I \cup M$. Finally, closedness of $\pi[M_\beta]$ above $\pi(\beta)$ is seen by inspection of Definitions 5.1, 5.2 of [14], Definition 3.9, and closedness of $\{\alpha_\pi^+\} \cup \pi[M]$ above α follows from the choice of $\pi(\beta)$. \square

4 Isominimal realization

Definition 4.1 Let $P \neq \emptyset$ be finite such that $P_a := \{b_1, \dots, b_r, a\} \dot{\cup} P$ is a respecting forest of order 2 over the language $(0; \leq, \leq_1, \leq_2)$, where the constant 0 does not need to be interpreted, $r \geq 0$,

$$b_1 <_2 \dots <_2 b_r <_2 a < P, \quad \text{and}$$

$$a <_1 \max(P) \quad \text{if } a > 0.$$

Suppose that $\alpha = 0$ if $a = 0$ and otherwise $\alpha < 1^\infty$ such that $\text{Pred}_2(\alpha) = \{\beta_1, \dots, \beta_r\}$ for ordinals $\beta_1 < \dots < \beta_r$.

1. A mapping $c_\alpha : P_a \rightarrow \mathcal{C}_2$ is called an α -covering of P_a if and only if $c_\alpha(b_i) = \beta_i$ for $i = 1, \dots, r$, $c_\alpha(a) = \alpha$, and $\text{Im}(c_\alpha)$ is a cover of P_a in \mathcal{C}_2 .
2. An α -covering c_α of P_a is called an α -isomorphism of P_a if $\text{Im}(c_\alpha)$ is isomorphic to P_a .
3. c_α is called an isominimal realization of P_a above α if and only if it is an α -covering that is \leq_{pw} -minimal among all α -coverings of P_a .
4. An α -covering c_α is called convex if $\text{tc}(\beta)$ is convex for all $\beta \in \text{Im}(c_\alpha)$.

Let $\alpha \in \text{TC} \cup \{()\}$ and M_α be (essentially) closed above α . Setting $\alpha := o(\alpha)$, $\text{Pred}_2(\alpha) =: \{\beta_1, \dots, \beta_r\}$, we define the (respecting) forest associated with M_α to be P_a , where $P := o[M_\alpha]$, $a := \alpha$, $b_i := \beta_i$ for $i = 1, \dots, r$, and $P_a := \{b_1, \dots, b_r, a\} \dot{\cup} P$.

Remark. For any respecting forest P_a of order 2, as in the above definition, there exists a convex α -covering c_α : we may simply choose the proof theoretic ordinal of a theory ID_N for a suitable index $N < \omega$ (setting $\text{ID}_0 := \text{PA}$), which provides a sufficiently long $<_2$ -chain to cover P_a .

Theorem 4.2 Let $\alpha \in \text{TC} \cup \{()\}$ and M_α be closed above α with associated forest P_a . Then the identity is the unique isominimal realization of P_a above α .

Proof. We argue by induction on the cardinality of M_α . Consider $\beta \in M_\alpha$ such that $\beta := o(\beta)$ is the largest immediate $<_1$ -successor of α in $o[M_\alpha]$, and let τ be the chain associated with β if $\alpha \subseteq \beta$ and with α otherwise. We obtain the partitioning

$$M_\alpha = M_0 \cup \{\beta\} \cup M_\beta,$$

where $M_0 := \{\delta \in M_\alpha \mid \delta <_{\text{TC}} \beta\}$, and observe that $\alpha \subseteq \delta$ for all $\delta \in M_0$, as is seen from Proposition 5.10 (and the remark thereafter) of [14], that M_0 is closed above α unless $M_0 = \emptyset$, that M_β is closed above β , and that

$$\text{lh}(\beta) = \max(\text{o}[M_\beta]), \quad (4)$$

and define C to be the chain of tracking chains of consecutively greatest immediate $<_1$ -successors from β to $\text{lh}(\beta)$ through $\text{o}[M_\beta]$. Let c_α be an α -covering of P_α and set $\gamma := \text{tc}(\gamma)$ where $\gamma := c_\alpha(\beta)$. For convenience we define

$$\text{vc}_\alpha := \text{tc} \circ c_\alpha \circ \text{o}.$$

Case 1: β extends α by $\beta_{n+1,1} =_{\text{NF}} \eta + \tau_{n+1,1}$. If $\eta > 0$, by closedness we either have $\alpha^\frown(\eta) \in M_\alpha$, or $n > 0$, $\eta = \tau_{n,m_n} \in \mathbb{E}^{>\tau_n}$, and the extension of α by the ν -index $\mu_{\tau_{n,m_n}}$ at $(n, m_n + 1)$ is an element of M_α . Then M_0 is non-empty and the i.h. applies to α and M_0 .

We now show that c_α is pointwise greater than or equal to the identity. Without loss of generality we may assume that the restriction of c_α to M_0 is the identity, that $\gamma = c_\alpha(\beta)$ is α - \leq_1 -minimal, and that $\gamma \leq \beta$. In the case $\gamma = \beta$ we directly apply the i.h., otherwise we have $\gamma = \alpha^\frown(\eta + \xi)$ for some $\xi \in (0, \tau_{n+1,1})$ and set $\sigma := \text{end}(\xi) = \text{end}(\gamma_{n+1,1})$. Now straightforward translation from γ to β leads to a contradiction with the i.h. for β and M_β , since

$$\log((1/\sigma^*) \cdot \sigma) < \log((1/\tau_{n+1}^*) \cdot \tau_{n+1,1}),$$

where σ^* is the $n+1$ -th unit of γ according to Definition 5.1 of [7], as γ and β have the same $<_2$ -predecessors.

Case 2: Otherwise. Then β either extends α by β_{n,m_n+1} (where $m_n \geq 1$), which by minimality of β and closedness satisfies $\beta_{n,m_n+1} \in \mathbb{P}$, and in which case we set $(i, j) := (n, m_n)$, or we have $\alpha \not\subseteq \beta$, so that according to Proposition 5.10 of [14] and closedness $(i, j+1) := \text{bp}(\alpha, \beta)$ exists with $\beta_{\upharpoonright i, j+1} = \beta$, and with $\alpha = \alpha_{\upharpoonright i, j+1}$ in the case $\chi^{\tau_{i,j}}(\tau_{i,j+1}) = 0$, while $\alpha = \text{me}(\alpha_{\upharpoonright i, j+1})$ if $\chi^{\tau_{i,j}}(\tau_{i,j+1}) = 1$. We then observe that $\beta_{i,j+1} = \alpha_{i,j+1} + \rho$ for some $\rho \in \mathbb{P}$. In both cases for β we have $\alpha \neq ()$ and $\tau := \tau_{i,j} \in \mathbb{E}^{>\tau'_{i,j}}$. If $\text{cml}(\beta)$ exists we set $(r, s) := \text{cml}(\beta)$, otherwise we let $(r, s) := (i, j)$.

As in Case 1, if the set M_0 is non-empty, we may apply the i.h. straightforwardly to see that the identity is the unique isominimal realization of M_0 above α . Note that the set $\{\beta\} \cup M_\beta$ is closed above α , and thus it suffices to show the claim for this set. To this end, assume c_α to be an α -covering of $\{\beta\} \cup M_\beta$. Without loss of generality we may assume that $\gamma = c_\alpha(\beta)$ is α - \leq_2 -minimal and less than or equal to $\text{lh}(\beta)$.

Claim 4.3 *We may assume that γ is of the form $\beta[\nu]$ for some $\nu \leq \beta_{i,j+1}$.*

Proof of Claim 4.3. We consider the following two cases.

Case A: $\beta_{i,j+1} = \mu_\tau$ and $\text{cml}(\beta)$ does not exist. Then M_β consists of proper extensions of β only. Moreover, setting $\alpha' := \alpha_{\upharpoonright i, j}$ the set $\{\beta\} \cup M_\beta$ is closed above α' and consists of proper extensions of α' only. We consider the case where γ does not extend α' in one step by a ν -index $\nu \leq \mu_\tau$. Note that while $\alpha <_{\text{TC}} \gamma$, by Lemma 5.11 of [14] we have $\alpha' \not\subseteq \gamma$ since $\alpha' := \text{o}(\alpha') \leq \alpha < \gamma \leq \text{lh}(\beta)$, which entails $c_\alpha(\text{lh}(\beta)) \leq \text{lh}(\beta)$. The α - \leq_2 -minimality of γ implies that γ is even α' - \leq_2 -minimal. Let $c_{\alpha'}$ be the appropriate restriction of c_α to become a α' -covering of $\{\beta\} \cup M_\beta$.

Writing $\gamma = (\gamma_1, \dots, \gamma_l)$, where $\gamma_r = (\gamma_{r,1}, \dots, \gamma_{r,k_r})$ for $r = 1, \dots, l$, according to Proposition 3.5 and our assumptions we have $k_l = 2$, $l^* <_{\text{lex}} (i, j)$, and $\sigma := \text{end}(\gamma_{l,1}) \in (\tau', \tau)$. Since $\gamma \not\subseteq \text{me}(\alpha')$ by our assumptions, $\text{cml}(\gamma_{\upharpoonright l,1})$ therefore does not exist, so that the tracking chains of image elements of c_α greater than γ are extensions of γ , whence by Theorem 3.7 we may assume that γ is of the form $\alpha'^\frown(\sigma)$.

If $\pi_{\sigma, \tau}^{-1}(\lambda^\sigma) < \lambda_\tau$, straightforward upward base transformation by $\pi_{\sigma, \tau}^{-1}$ and translation from γ to β yields a contradiction with the i.h. for β and M_β . Otherwise we have

$$\tau' < \sigma \leq \bar{\tau} \leq \text{mp}(M_\beta) =: \rho < \tau$$

by closedness. Let $\xi \in M_\beta$ be $<_{\text{TC}}$ -minimal such that $\text{par}_{M_\beta}(\xi) = \rho$, so that M_ξ is closed above ξ and only consists of extensions of ξ as $\text{cml}(\beta)$ and hence also $\text{cml}(\xi)$ do not exist.

If $\sigma < \rho$, we immediately obtain a contradiction with the i.h. for $\{\delta\} \cup M_\delta$ above α' where $\delta := \alpha'^\frown(\rho)$ and M_δ is the translation of M_ξ to δ .

In the remaining case, where $\sigma = \bar{\tau} = \rho$, by closedness we must have $\text{db}(M_\beta) = 0$, and the same translation of M_ξ to $\delta = \alpha'^\frown(\bar{\tau})$ results in a set M_δ such that $\{\delta\} \cup M_\delta$ is closed above α' , for which the appropriate restriction of $c_{\alpha'}$ contradicts the i.h., since this covering does not use the maximal branch of M_δ .

Case B: Otherwise. If $\text{cml}(\beta) = (r, s)$ exists, we have $(r, s) \leq_{\text{lex}} (i, j)$, otherwise we must have $\beta_{i,j+1} < \mu_\tau$ and $(r, s) = (i, j)$. In either case, we then have $(r, s) <_{\text{lex}} (i, j)$ if and only if $\beta_{i,j+1} = \mu_\tau$, and due to closedness we have

$$\delta := \beta_{\upharpoonright_{r,s+1}}[\mu_{\tau,r,s}] \in M_\beta \quad \text{and} \quad \beta < o(\delta) =: \delta.$$

Setting $\alpha' := \alpha_{\upharpoonright_{r,s}}$, $\alpha' := o(\alpha')$, and $M' := \{\zeta \in M_\beta \mid \delta \leq_{\text{TC}} \zeta\}$, we observe that M' is closed above α' and that the restriction c' of c_α to α' and M' is an α' -covering, wherefore the i.h. applies to reveal that the image of c' is pointwise greater than or equal to the identity.

Let ζ be the least element in C such that $\beta \not\leq_2 \zeta$. Since $\gamma = c_\alpha(\beta)$ is $\alpha \leq_2$ -minimal and $\gamma \leq \text{lh}(\beta)$, the assumption $\beta < \gamma$ implies that $\beta \not\leq_2 \gamma$ and that $\gamma \leq_1 \text{lh}(\beta) = c_\alpha(\text{lh}(\beta))$. Under this assumption we may modify c_α to be the identity on $\{\beta\} \cup o[M_\beta] \cap \zeta$, resulting in an α -covering pointwise below c_α . As the i.h. applies in the case $\gamma = \beta$, we may therefore assume that $\gamma < \beta$. Since $\text{lh}(\beta) \leq c_\alpha(\text{lh}(\beta))$ as shown above, we have $\gamma <_1 \text{lh}(\beta)$, and thus we may assume that γ is $\alpha \leq_2$ -minimal such that $\gamma <_1 \beta$, implying the claim.

This concludes the proof of Claim 4.3. \square

Claim 4.4 *We may assume that the image $V := \text{vc}_\alpha[M_\subseteq]$ of the \leq_{TC} -initial segment*

$$M_\subseteq := \{\zeta \in I(\beta) \cup M_\beta \mid \beta_{\upharpoonright_{r,s+1}} \subseteq \zeta\}$$

of $\{\beta\} \cup M_\beta$ consists of extensions of $\text{vc}_\alpha(\beta_{\upharpoonright_{r,s+1}})$ only.

Proof of Claim 4.4. We set $\beta^\# := \beta_{\upharpoonright_{r,s+1}}$ and $\gamma^\# := \text{vc}_\alpha(\beta^\#)$. Note that $\gamma^\# = \beta^\# = \alpha_{\upharpoonright_{r,s+1}}$ in the case $(r, s) <_{\text{lex}} (i, j)$. Let V_1, V_2 be the partitioning of V into extensions of $\gamma^\#$ and tracking chains ζ such that $\gamma^\# \not\leq \zeta$, respectively, and let M_1, M_2 be the corresponding preimages. Let us assume that $V_2 \neq \emptyset$. Due to Lemma 5.11 of [14] we have $V_1 <_{\text{TC}} V_2$ and hence also $M_1 <_{\text{TC}} M_2$. Note that there does not exist any \leq_2 -connection from M_1 into M_2 as there does not exist any such connection from V_1 into V_2 . We consider the decomposition of $o[M_2]$ into \leq_1 -connectivity components, writing

$$M_2 = \bigcup_{p=1}^q \{\xi_p\} \cup M_{\xi_p},$$

where $\beta <_{\text{TC}} \xi_1 <_{\text{TC}} \dots <_{\text{TC}} \xi_q$. Then the ordinals $\xi_p := o(\xi_p)$, $p = 1, \dots, q$, are $\alpha' \leq_2$ -minimal, where $\alpha' := o(\alpha')$ and $\alpha' := \alpha_{\upharpoonright_{r,s}}$. Hence, by Proposition 3.5, each ξ_p is of a form $\zeta_p \frown (\xi_{p,k_p,1}, \dots, \xi_{p,k_p,l_{p,k_p}})$, where $l_{p,k_p} \leq 2$ and $\text{end}(\xi_{p,k_p,1}) < \tau_{r,s}$. Clearly, $l_{p,k_p} = 1$ for $p = 2, \dots, q$, and we may assume that also $l_{1,k_1} = 1$, since the case $l_{1,k_1} = 2$ is handled similarly, as $o(\xi_1 \frown (\xi_{1,k_1,1}))$ then must be $\alpha' \leq_2$ -minimal as well. Note that we have $\beta^\# \subseteq \zeta_p$ and that each $\text{cml}(\xi_p)$ would have to satisfy $\text{cml}(\xi_p) <_{\text{lex}} (r, s)$ and therefore does not exist for $p = 1, \dots, q$. Hence, each M_{ξ_p} is closed above ξ_p and consists of extensions of ξ_p only.

We may thus modify the restriction of c_α to $o[M_\subseteq]$ on M_2 by the appropriate translations of the components $\{\xi_p\} \cup M_{\xi_p}$ to successively append $<_1$ -branches to the greatest common $<_1$ -predecessor in $o[V_1]$ of the ordinals in $o[V_2]$, which is possible due to property 9 in the remark after Proposition 5.10 of [14]. This modification results in a covering that is pointwise less than or equal to c_α .

This concludes the proof of Claim 4.4. \square

Case 2.1: $\beta_{i,j+1} < \mu_\tau$. Then we are in the scenario of Case B above.

Subcase 2.1.1: $\text{cml}(\beta)$ does not exist. Then M_\subseteq is closed above β , the i.h. applies, and we are done.

Subcase 2.1.2: Otherwise. Then we have $\text{cml}(\beta) = (i, j)$; hence $\beta^\# = \beta$, $\chi^\tau(\rho) = 1$ where $\rho := \text{end}(\beta_{i,j+1})$, and setting $\xi := \text{me}(\beta) \in M_\subseteq$ we have $\beta <_2 o(\xi) =: \xi$ and ξ is the immediate predecessor of ζ in C , where ζ is defined as in the above Case B. Note that $\beta \not\leq \zeta := \text{tc}(\zeta)$ and $\gamma <_2 c_\alpha(\xi) <_1 c_\alpha(\text{lh}(\beta))$ in this situation. According to Claim 4.4, V consists of extensions of γ only, containing $\text{tc}(c_\alpha(\xi))$. Now define M'_\subseteq to be the translation of M_\subseteq from β to

$$\beta' := \beta[\beta_{i,j+1} + \rho \cdot \omega]$$

with the additional tracking chain $\text{tc}(\text{lh}_2(\beta'))$, where $\beta' := o(\beta')$. Then $M'_\subseteq - \{\beta'\}$ is closed above β' and contains less elements than M_β , since $\{\beta\} \cup M_\beta$ is closed above α . If $\gamma = \beta[\nu]$ for some $\nu < \beta_{i,j+1}$, $\nu =_{\text{NF}} \nu' + \nu_0$, we

must have $\chi^\tau(\nu_0) = 1$, since $c_\alpha(\xi) <_1 c_\alpha(\text{lh}(\beta))$, where $\text{tc}(c_\alpha(\text{lh}(\beta)))$ is not reachable by extension of γ . Let V' be the translation of V from γ to

$$\gamma' := \beta[\nu' + \nu_0 \cdot \omega]$$

with the additional tracking chain $\text{tc}(\text{lh}_2(\gamma'))$, where $\gamma' := o(\gamma')$. Translating V' from γ' to β' then gives rise to a β' -covering of $o[M'_\subseteq - \{\beta'\}]$ that contradicts the i.h.

Case 2.2: $\beta_{i,j+1} = \mu_\tau$.

Subcase 2.2.1: $\text{cml}(\beta)$ does not exist. Then we are in the situation of the above Case A, where $M_\beta \subseteq M'_\subseteq$, and the assumption $\gamma = \beta[\nu]$ for some $\nu < \beta_{i,j+1}$ leads to a contradiction with the i.h. by straightforward translation of $V = \text{vc}_\alpha[\{\beta\} \cup M_\beta]$, which according to Claim 4.4 consists of extensions of γ only, from γ up to β .

Subcase 2.2.2: $\text{cml}(\beta) = (r, s)$ exists. Here Claim 4.4 applies with $\beta^\# \subsetneq \beta$ since $(r, s) <_{\text{lex}} (i, j)$, cf. the above Case B. Note that we therefore have $\gamma^\# = \beta^\# = \alpha|_{r,s+1}$, and setting $\xi := \text{me}(\beta) = \text{me}(\beta^\#)$ we have $\beta^\# := o(\beta^\#) <_2 o(\xi) =: \xi$, and ξ is the immediate predecessor of ζ in C , where ζ is now defined to be the least element of C such that $\beta^\# \not\leq_2 \zeta$. Note that $\beta^\# \not\leq \zeta := \text{tc}(\zeta)$ and

$$\beta^\# <_2 c_\alpha(\xi) <_1 c_\alpha(\text{lh}(\beta)),$$

showing that $c_\alpha(\xi) \in V$ while $\beta^\# \not\leq \text{tc}(c_\alpha(\text{lh}(\beta)))$. Now define M'_\subseteq to be the translation of M'_\subseteq from $\beta^\#$ to

$$\beta^+ := \beta^\#[\alpha_{r,s+1} + \rho \cdot \omega],$$

where $\rho := \text{end}(\alpha_{r,s+1}) \in \mathbb{P}$, with the additional tracking chain $\text{tc}(\text{lh}_2(\beta^+))$, where $\beta^+ := o(\beta^+)$. Let β' be the image of β under this translation, i.e. the $<_{\text{TC}}$ -minimal element of M'_\subseteq , and note that $\text{cml}(\beta')$ does not exist. Assuming that $\gamma = \beta[\nu]$ for some $\nu < \beta_{i,j+1}$ such that $\alpha < \gamma$, let γ' be the image of γ under the same translation from $\beta^\#$ to β^+ , i.e., γ' results from γ by replacement of the index at $(r, s+1)$ by $\alpha_{r,s+1} + \rho \cdot \omega$, and redefine α' in the same way. Then M'_\subseteq is closed above α' and contains less elements than $\{\beta\} \cup M_\beta$. Let $c_{\alpha'}$, where $\alpha' := o(\alpha')$, result from c_α by the same index replacement that performs the translation from $\beta^\#$ to β^+ , so that it maps the elements of its domain M'_\subseteq to the corresponding translated image elements of c_α while fixing $\text{lh}_2(\beta^+)$. Then setting $\xi' := o(\xi')$, where ξ' results from translating ξ , we have

$$\beta^+ <_2 c_{\alpha'}(\xi') <_1 \text{lh}_2(\beta^+),$$

and $c_{\alpha'}$ is an α' -covering of M'_\subseteq contradicting the i.h.

We therefore must have $\nu = \beta_{i,j+1}$, whence the claim for α and M_α follows from the i.h. for β and M_β . \square

Remark. Note that any covering of an essentially closed set M extends to a covering of its closure \bar{M} under initial chains. Hence essentially closed sets are uniquely isominimally realized by the identity.

Theorem 4.5 *Let P_a be a respecting forest of order 2 as in the above definition, with a given convex α -covering c_α , and set $\alpha := \text{tc}(\alpha)$ if $\alpha > 0$, and $\alpha := ()$ if $\alpha = 0$. There exists a unique α -isomorphism i_α of P_a such that*

1. $i_\alpha[P]$ is closed under lh, lh_2 and
2. $\text{tc} \circ i_\alpha[P]$ is essentially closed above α .

Proof. We argue by induction on the cardinality of P . Note that property 1 follows from property 2 by Corollaries 5.13 and 5.9 of [14] and Lemma 3.4. Let $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,m_i})$, $1 \leq i \leq n$, $n \geq 0$, be the components of α . Let $P = \bigcup_{i=1}^{k+1} P_i$ be the partitioning of P into increasing $a \leq 1$ -connectivity components. Let Q be any of the P_i and set $q := \min(Q)$, i.e., in the case $a = 0$ the element q is the i -th \leq_1 -minimal element in P , and otherwise q is the i -th immediate $<_1$ -successor of a in P . Then the restriction of c_α to Q_a remains to be a convex α -covering, and we may assume that $\beta := c_\alpha(q)$ does not have any $<_2$ -predecessor in (α, β) , since otherwise we would obtain another convex α -covering of P_a by simply replacing β by such a $<_2$ -predecessor. The convexity of α furthermore implies that $\alpha \subseteq \beta := \text{tc}(\beta)$ where $\beta_i = (\beta_{i,1}, \dots, \beta_{i,k_i})$, $1 \leq i \leq l$, and the convexity of c_α implies that β is convex. Let τ be the associated chain. In the case where $k_l = 1$ we have $l > n$ and $l^* <_{\text{lex}} (n, m_n)$ due to the $\alpha \leq_2$ -minimality of β , and q does not have any $<_2$ -successor in P . If $k_l > 1$ then β is a principal chain, and due to the $\alpha \leq_2$ -minimality of β we either have $(l, k_l) = (n, m_n + 1)$ and $\beta_{l,k_l} = \mu_{\tau_n, m_n}$, or $l > n$, $k_l = 2$, $\beta_{l,2} = \mu_{\tau_{l,1}}$, and $l^* <_{\text{lex}} (n, m_n)$. In the cases where $l > n$ we may assume that $l = n + 1$ due to Theorem 3.7.

Now, if necessary, the i.h. is applied to P_q , defined as the substructure of P_a given by the union of the subset of $\{b_1, \dots, b_r, a, q\}$ matching the \leq_2 -predecessors of β with the set of elements of P that are $<_1$ -successors of q , and the appropriate restriction of c_α . We thus obtain (in the non-trivial case) a β -isomorphism i_β and define M_β to be the closure of $\text{tc} \circ i_\beta[Q^{>q}]$ under initial chains, so that M_β is either empty or closed above β , cf. Definition 3.12. Setting for convenience $b_{r+1} := a$ and $\beta_{r+1} := \alpha$, let σ be the base of β_i where b_i is the greatest $<_2$ -predecessor of q in P_a if such exists and $\sigma := 1$ otherwise. We now define the set $M := \{\beta\} \cup M_\beta$ of proper extensions of α and consider the following two cases.

Case 1: q does not have any $<_2$ -successor in P . Here we may apply κ -index minimization above σ in M at β , cf. Definition 3.14 and Theorem 3.15, and set $M_q := \kappa[M]$, $\beta_q := \kappa(\beta)$, $\beta_q := o(\beta')$, and $\xi_q := \kappa\text{-idx}$.

Case 2: Otherwise, base minimization above σ in M at β applies, cf. Definition 3.20 and Theorem 3.21, and we set $M_q := \{\alpha_\pi^+\} \cup \pi[M]$, $\beta_q := \pi(\beta)$, $\beta_q := o(\beta_q)$, and $\xi_q := \pi\text{-idx}$.

Now M_q is closed above α , and using straightforward translation we can define the mapping i_α on P . We have κ -indices ξ_{q_i} for $i = 1, \dots, k+1$ where $q_i = \min(P_i)$ and define $\xi_i := \sum_{j=1}^i \xi_{q_j}$ for $i = 1, \dots, k+1$. Changing the κ -index ξ_{q_i} to ξ_i at $(n+1, 1)$ in every chain in M_{q_i} for each i where it applies (i.e. there is no change in the case where β_{q_i} extends α directly by a ν -index), we obtain the image of i_α after omitting superfluous chains ending in κ -indices that do not match elements in P_a from the modified M_{q_i} . The image of P under $\text{tc} \circ i_\alpha$ is therefore essentially closed above α , as desired. \square

Theorems 4.2 and 4.5 now readily combine to the following main result on isomorphic copies of respecting forests of order 2 in \mathcal{C}_2 that are unique in being pointwise minimal among all coverings.

Corollary 4.6 *Every respecting forest P of order 2 (and hence every pure pattern of order 2) has a unique isominimal realization $i[P]$ in \mathcal{C}_2 . $i[P]$ is isomorphic to P , essentially closed, and hence closed under lh and lh₂.*

Isominimal realizations are therefore tight within \mathcal{C}_2 as there do not exist \leq_1 - nor \leq_2 -connections to elements of \mathcal{C}_2 that extend beyond the respective largest connections in the realization.

Corollary 4.7 (Ordinal notations) *Let $\alpha < 1^\infty$ and $M := \{\text{tc}(\alpha)\}^{\text{ecl}}$ its essential closure. Then the respecting forest P associated with M together with a marker for the element matching α provides a pattern notation for α . This notation is of least cardinality possible.*

Proof. Let Q be a respecting forest of order 2, of which the unique isominimal realization $c[Q]$ within \mathcal{C}_2 contains α . $c[Q]$ is essentially closed. Inspection of Definitions 3.11 and 3.13 shows that we (must) make a choice (choosing a normal form) when performing a closure, but in a way that adds as few new elements as possible. Hence Q must have at least as many elements as P . \square

Together with the obvious, elementary recursive comparison relations, we therefore obtain an elementary recursive notation system for the ordinal 1^∞ .

Corollary 4.8 *The union of all isominimal realizations of respecting forests of order 2 comprises the initial segment 1^∞ of the ordinals, characterizing the core of \mathcal{R}_2 .*

Proof. By Theorem 7.4 of [7] we know that the arithmetical characterization $\mathcal{C}_2 = (1^\infty; \leq, \leq_1, \leq_2)$ coincides with the structure $\mathcal{R}_2 \upharpoonright_{1^\infty}$, where \leq_1 and \leq_2 are defined as Σ_1 - and Σ_2 -elementary substructurehood, respectively. $\text{Core}(\mathcal{R}_2)$ is by definition the union of all isominimal copies of finite isomorphism types of \mathcal{R}_2 . Corollary 4.6 shows that each respecting forest of order 2 is a finite isomorphism type of \mathcal{C}_2 , which by Theorem 7.4 of [7] is a finite isomorphism type of \mathcal{R}_2 with coinciding isominimal realizations, hence $\text{Core}(\mathcal{R}_2) = 1^\infty$. \square

We finally come to a statement regarding the combinatorial strength of respecting forests of order 2. Recall the enumeration function κ of the \leq_1 -minimal ordinals in \mathcal{C}_2 , cf. its extension from [1] for the segment ε_0 to 1^∞ in Definition 4.4 of [7] and Section 4 of [14].

Corollary 4.9 *Denote the notation for an ordinal $\gamma < 1^\infty$ given in Corollary 4.7 by $P(\gamma)$. Let $\alpha < \beta < 1^\infty$. Then there does not exist any covering of $P(\kappa_{\omega^\beta})$ into $P(\kappa_{\omega^\alpha})$. Hence any infinite descending sequence of ordinals below 1^∞ produces an infinite bad sequence of respecting forests of order 2 with respect to coverings.* \square

Together with Carlson's result that respecting forests of order 2 are well-quasi-ordered with respect to coverings, cf. [5], we obtain the independence of this wqo-result of the theory KPl_0 , since as seen above, the well-quasi orderedness would imply $TI(1^\infty)$, i.e. transfinite induction up to 1^∞ , i.e. the proof-theoretic ordinal of KPl_0 (equivalently $\Pi_1^1 - CA_0$). On the other hand, we have seen by Theorem 7.4 of [7] that $TI(1^\infty)$ suffices to show that every finite substructure of \mathcal{R}_2 has a covering contained in 1^∞ .

5 Conclusion

The structure \mathcal{C}_2 , which arithmetically characterizes the structure \mathcal{R}_2 of pure elementary patterns of resemblance of order 2 up to 1^∞ as proven in [7], was shown in [14] to be elementary recursive. Here we have established mutual elementary recursive order isomorphisms between classical ordinal notations and pattern notations, showing that pattern notations based on pure Σ_2 -elementarity characterize the proof theoretic ordinal 1^∞ of the fragment $\Pi_1^1 - CA_0$ of second order number theory, or equivalently, the set-theoretic system KPl_0 , which axiomatizes limits of admissible universes (i.e. models of $KP\omega$, Kripke-Platek set theory with infinity).

We have seen that the finite isomorphism types of \mathcal{C}_2 , hence of \mathcal{R}_2 , comprise (up to isomorphism) the class of respecting forests of order 2, cf. [3] and [4]. We have shown that the union of isomimal realizations of respecting forests of order 2 is indeed the core of \mathcal{R}_2 and is to equal the proof-theoretic ordinal of KPl_0 . As a corollary we have proven that the well-quasi orderedness of respecting forests with respect to coverings, which was shown by Carlson in [5], implies (in a weak theory) transfinite induction up to the proof-theoretic ordinal 1^∞ of KPl_0 .

We expect, as mentioned in [14], that the approaches taken here and in our treatment of the structure \mathcal{R}_1^+ , see [12] and [13], will naturally extend to an analysis of the structure \mathcal{R}_2^+ and possibly to structures of patterns of higher order. A subject of ongoing work is to verify our claim that the core of \mathcal{R}_2^+ matches the proof-theoretic strength of a limit of KPI-models, which in turn axiomatize admissible limits of admissible universes.

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